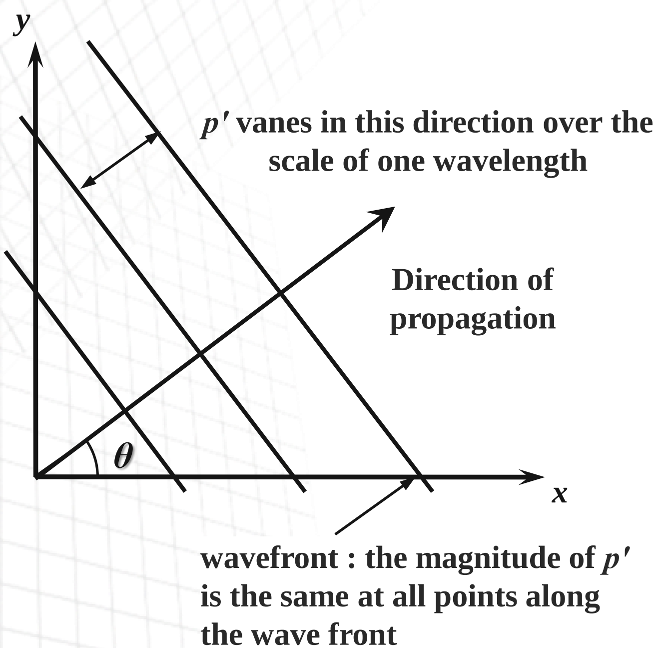
❖ The Ray theory equations

● Plane wave of homogeneous medium

- A plane wave has the distinctive property that its strength and direction of propagation do not vary as it propagates through a homogeneous medium

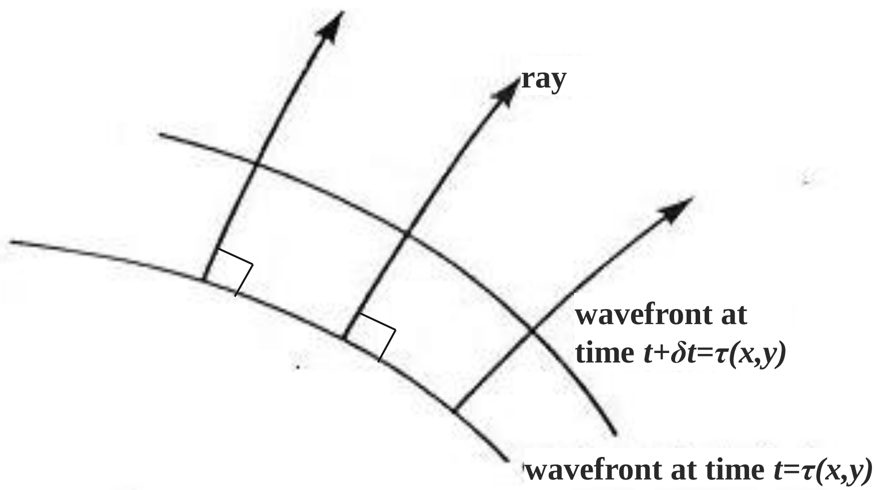


$$p' = Ie^{i\omega(t-\tau(x,y))}$$

$$\tau(x,y) = x/c\cos\theta + y/c\sin\theta$$

❖ The Ray theory equations

- On the other hand, a wave propagating through a region with a slowly varying sound speed will have a slightly curved wave front
- Rays are defined to be curves which are always normal to the wavefront
- Rays are useful concept if $L \gg \lambda$, i.e. high frequency waves. Then the only effect of variation in the sound speed is that speed of propagation of the wavefront along the ray.



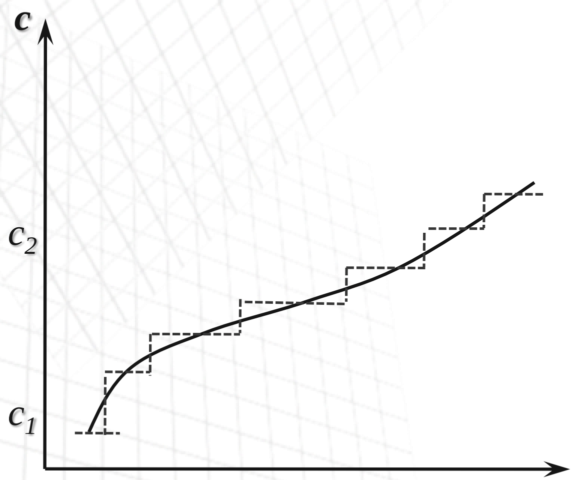
❖ The Ray theory equations

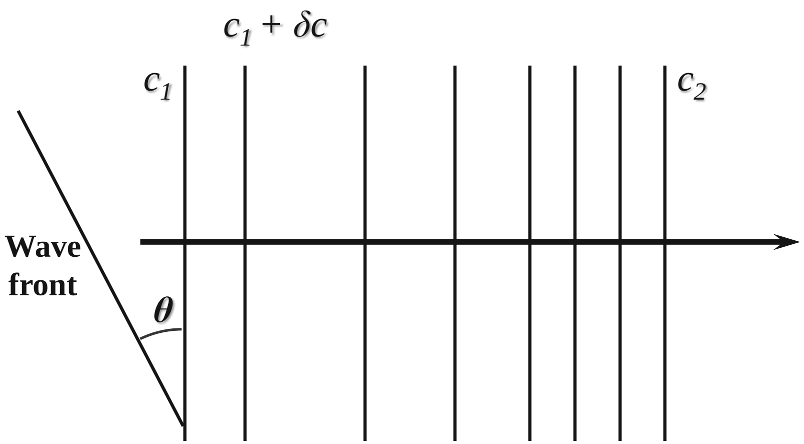
$$p'(x,y,t) = I(x,y)e^{i\omega(t-\tau(x,y))}$$

- (Note)
  - (i) ***I*** varies because of gradual changes in the sound speed.
  - (ii) The variation of ***I*** occurs over the length scale over which the sound speed varies (generally long compared to wavelength)
  - (iii) The phase is a function of position and accounts for the variation p’ over the scale of wavelength
  - (iv) If, L, the scale on which the sound speed varies, is MUCH greater than the wave length, then I is slowly varying compared to ***τ*** and the moving surface, ***t- τ (x,y) = constant*** to be wavefronts

❖ The Ray theory equations

- Transmission through the stratified medium
  - A continuous variation can be considered as an approximation of *N* jumps



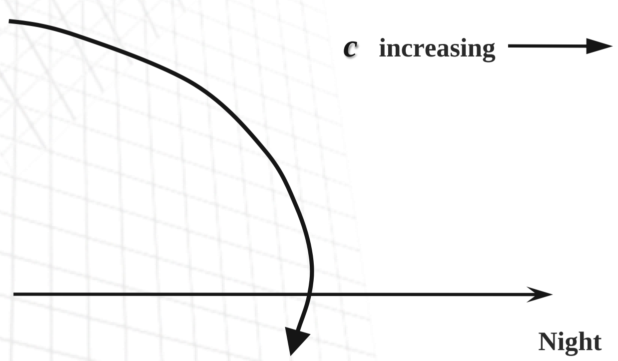


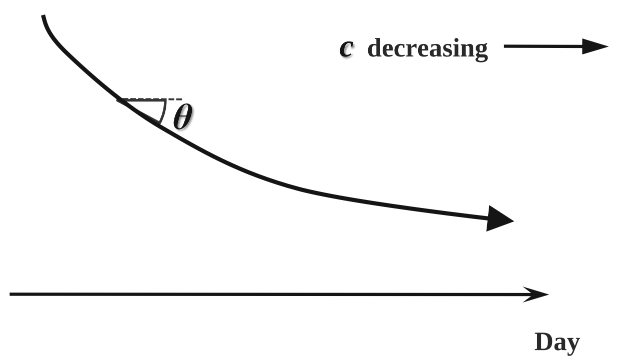
❖ The Ray theory equations

- When a plane wave propagates from the one medium to another with a different sound speed, Snell's law is valid in transmission.

$$\frac{\sin\theta}{c} = \text{constant} \qquad c \le \frac{c_0}{\sin\theta_0}$$

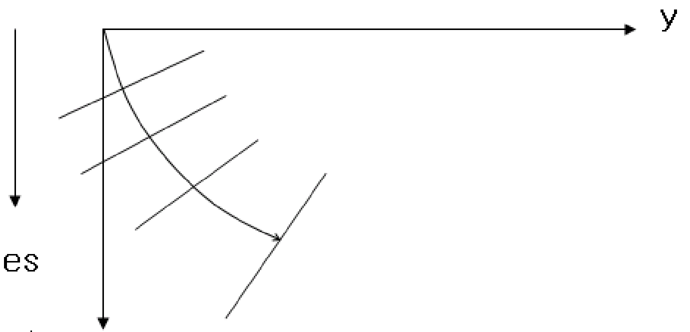
- Snell's law can be used to determine the rays paths and hence to discover where sound is heard
- The rays bent to the direction of decreasing of *'c'*





❖ The Ray theory equations

- (Example) the ocean



- Snell's Law : $$\frac{\sin\theta_0}{C_0} = \frac{\sin\theta}{C}$$

- where $$\sin\theta = \frac{dy}{\sqrt{dx^2+dy^2}} = \frac{\frac{dy}{dx}}{\{1+(\frac{dy}{dx})^2\}^{1/2}} = \sin\theta_0 \frac{C}{C_0} = \sin\theta_0(1+\alpha x)$$

❖ The Ray theory equations

- (Example) the ocean

$$\therefore \left(\frac{dy}{dx}\right)^2 = \left(\left(\frac{dy}{dx}\right)^2 + 1\right)(1+\alpha x)^2 \sin^2\theta_0)$$

$$\frac{dy}{dx} = \pm \frac{(1+\alpha x)\sin\theta_0}{\{1-(1+\alpha x)^2\sin^2\theta_0\}^{1/2}}$$

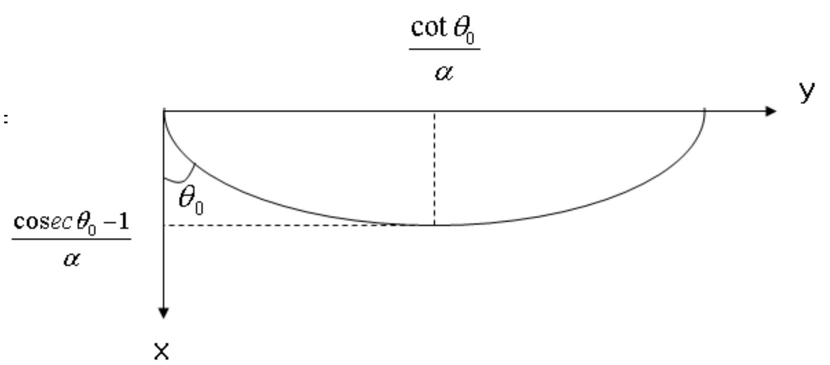
$$y = \pm\int_0^x \frac{(1+\alpha x)\sin\theta_0 dx}{\{1-(1+\alpha x)^2\sin^2\theta_0\}^{1/2}}$$

$$= \mp\frac{1}{\alpha\sin\theta_0}\{1-(1+\alpha x)^2\sin^2\theta_0\}^{1/2} \pm \frac{\cos\theta_0}{\alpha\sin\theta_0}$$

- $x = 0, y = \mp\frac{1}{\alpha\sin\theta_0}(1-\sin^2\theta_0)^{1/2} = \mp\frac{\cos\theta_0}{\alpha\sin\theta_0}$

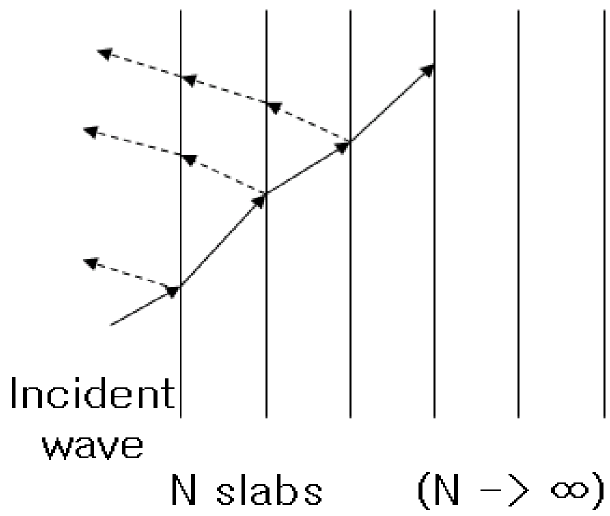
$$\left(y - \frac{\cot\theta_0}{\alpha}\right)^2 + \frac{1}{\alpha^2\sin^2\theta_0}(1-(1+\alpha x)^2\sin^2\theta_0)$$

$$\left(y - \frac{\cot\theta_0}{\alpha}\right)^2 + \left(x + \frac{1}{\alpha}\right)^2 = \frac{\operatorname{cosec}^2\theta_0}{\alpha^2}$$



❖ The Ray theory equations

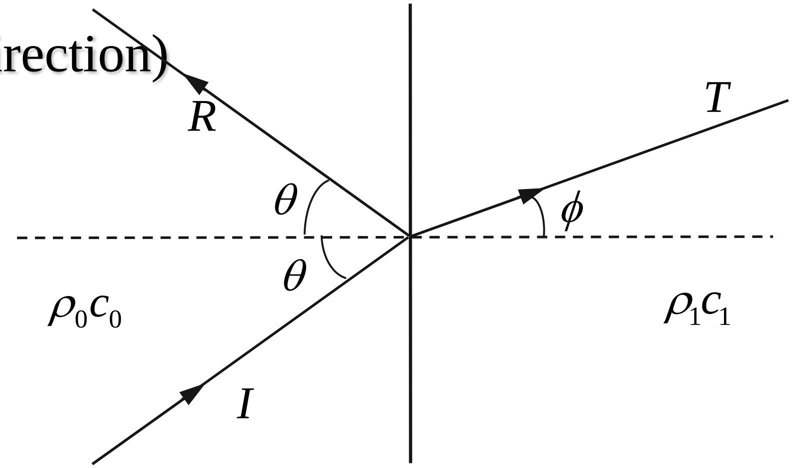
- Continuously varying medium
  - ① c is constant within each slab
  - ② C_{max}, C_{min} : $\Delta C \sim \dfrac{C_{max}-C_{min}}{N}$



❖ The Ray theory equations

- Reflection Coefficient of Plane wave at the interface with sound speed and densities $\boldsymbol{\rho_0}$, $\boldsymbol{\rho_1}$

The pressure & velocity (normal direction) should be continuous @ interface

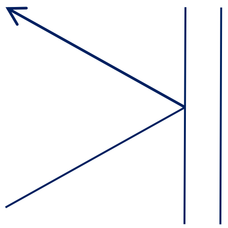


$$\frac{R}{I} = \frac{\dfrac{\rho_1 c_1}{\cos\phi} - \dfrac{\rho_0 c_0}{\cos\theta}}{\dfrac{\rho_1 c_1}{\cos\phi} + \dfrac{\rho_0 c_0}{\cos\theta}}$$

$$\cos\phi = \left(1 - \sin^2\theta \frac{c_1^2}{c_0^2}\right)^{1/2} \qquad \left|\frac{R}{I}\right| = \Delta c \sim \frac{1}{N}$$

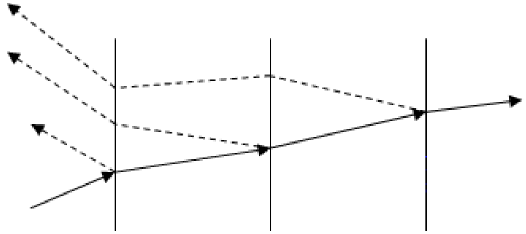
❖ The Ray theory equations

- compact layer ($L \ll \lambda$)



  - No phase difference
  - Transmission & Reflection depends only on the net change in acoustic properties 'across' the layer. (Neglect the change of acoustic properties within the layer)

- non - compact layer ($L \gg \lambda$)



  - Large phase difference between incident and reflected waves.
  - Energy reflected from each interface

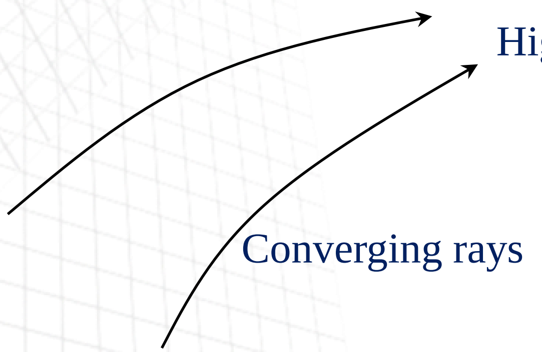
$$|I|^2\left|\frac{R}{I}\right|^2 = \frac{|I|^2}{|N|^2} \qquad N - \text{interfaces} \qquad E_R \sim \frac{|I|^2}{N}$$

❖ The Ray theory equations

- If $N \to \infty$

  " the actual continuous variation in sound speed and no energy is reflected . " ($E_R \to 0$)

- <u>No reflection energy for a high frequency sound ray propagation through a medium</u> in which the sound speed varies continuously .
- Energy flux is constant along the ray tube of sound propagating through a medium in which ρ_0 and c vary slowly ($L \gg \lambda$)



"Ray theory can be used to determine the level of sound heard"

❖ A more rigorous derivation of Ray theory

● Derivation of Ray theory

- Conservation of mass states that

$$\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial x}(\rho_0 u) - \frac{\partial}{\partial t}(\rho_0 v)$$

- Momentum equation of each direction is

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} \qquad \frac{\partial v}{\partial t} = -\frac{1}{\rho_0}\frac{\partial p}{\partial y}$$

- These two equations can be used for substitution

$$\frac{1}{c^2}\frac{\partial^2 p'}{\partial t^2} = \rho_0 \frac{\partial}{\partial x}\left(\frac{1}{\rho_0}\frac{\partial p'}{\partial x}\right) + \rho_0 \frac{\partial}{\partial y}\left(\frac{1}{\rho_0}\frac{\partial p'}{\partial y}\right)$$

- Ray series is derived from the harmonic functions of time.

$$p'(x, y, t) = e^{i\omega(t - \tau(x,y))}\sum_{n=0}^{\infty}(i\omega)^{-n} I_n(x, y)$$

❖ A more rigorous derivation of Ray theory

- Rays are defined to be curves which are everywhere normal to the wavefront

$$\left(\frac{dX}{ds},\frac{dY}{dx}\right)=\frac{\text{grad }\tau}{\left|\text{grad }\tau\right|}$$

- From the derivatives of ray series, substitution of the correct form for the wave equation leads to

$$\sum_{n=0}^{\infty}(i\omega)^{-n}\left[(i\omega)^{2}\left\{\left(\frac{\partial\tau}{\partial x}\right)^{2}+\left(\frac{\partial\tau}{\partial y}\right)^{2}-\frac{1}{c^{2}}\right\}I_{n}-i\omega\left\{\rho_{0}\frac{\partial}{\partial x}\left(\frac{1}{\rho_{0}}\frac{\partial\tau}{\partial x}I_{n}\right)+\rho_{0}\frac{\partial}{\partial y}\left(\frac{1}{\rho_{0}}\frac{\partial\tau}{\partial y}I_{n}\right)\right.\right.$$
$$\left.\left.+\frac{\partial\tau}{\partial x}\frac{I_{n}}{\partial x}+\frac{\partial\tau}{\partial y}\frac{I_{n}}{\partial y}\right\}+\rho_{0}\frac{\partial}{\partial x}\left(\frac{1}{\rho_{0}}\frac{\partial I_{n}}{\partial x}\right)+\rho_{0}\frac{\partial}{\partial y}\left(\frac{1}{\rho_{0}}\frac{\partial I_{n}}{\partial y}\right)\right]=0$$

❖ A more rigorous derivation of Ray theory

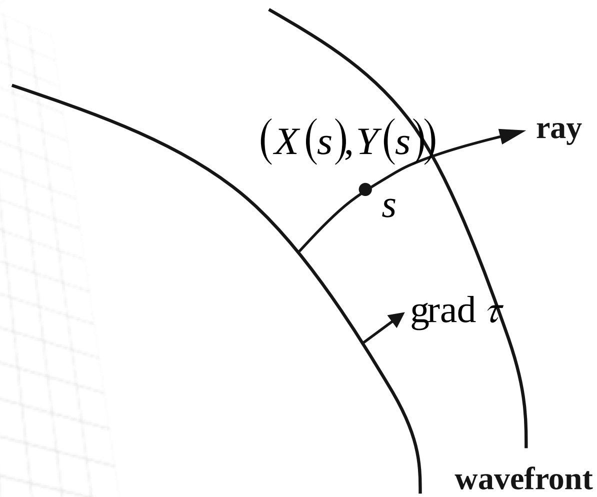
- First, the equation is to be true for all values of ω

$$\left(\frac{\partial\tau}{\partial x}\right)^2+\left(\frac{\partial\tau}{\partial y}\right)^2-\frac{1}{c^2}=0$$

'Eikonal Equation'

- The coefficient of ω must also vanish,

$$\rho_0\frac{\partial}{\partial x}\left(\frac{1}{\rho_0}\frac{\partial\tau}{\partial x}I_n\right)+\rho_0\frac{\partial}{\partial y}\left(\frac{1}{\rho_0}\frac{\partial\tau}{\partial y}I_n\right)+\frac{\partial\tau}{\partial x}\frac{I_n}{\partial x}+\frac{\partial\tau}{\partial y}\frac{I_n}{\partial y}=0$$



❖ A more rigorous derivation of Ray theory

- Using the chain rules of direction with regard to '***s***' direction, eikonal equation leads to the Snell's law

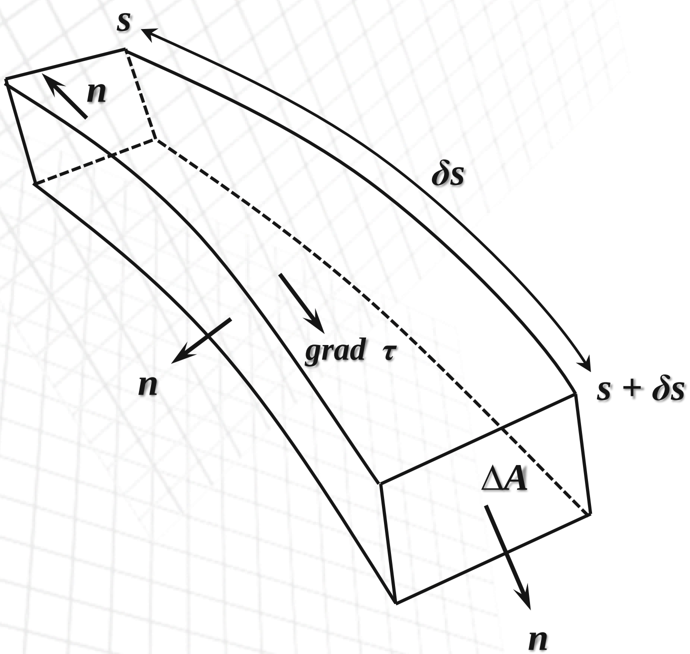
$$\frac{d}{ds}\left(\frac{1}{c}\frac{dY}{ds}\right)=\frac{d}{ds}\frac{\partial\tau}{\partial y}=c\left(\frac{\partial\tau}{\partial x}\frac{\partial}{\partial x}+\frac{\partial\tau}{\partial y}\frac{\partial}{\partial y}\right)\frac{\partial\tau}{\partial y},\qquad \frac{dY}{ds}=\sin\theta$$

- The intensity, $\boldsymbol{I_0}$, must satisfy the 2nd equation.

$$2\left(\frac{\partial\tau}{\partial x}\frac{I_0}{\partial x}+\frac{\partial\tau}{\partial y}\frac{I_0}{\partial y}\right)+I_0\left(\frac{\partial^2\tau}{\partial x^2}+\frac{\partial^2\tau}{\partial y^2}+\rho_0\frac{\partial\tau}{\partial x}\frac{\partial}{\partial x}\left(\frac{1}{\rho_0}\right)+\rho_0\frac{\partial\tau}{\partial y}\frac{\partial}{\partial y}\left(\frac{1}{\rho_0}\right)\right)=0$$

❖ A more rigorous derivation of Ray theory

- The solution to this equation is



$$I_0(s) = I_0(s_0)\exp\left[-\frac{1}{2}\int_{s_0}^{s}\left\{c\nabla^2\tau + \rho_0\frac{\partial}{\partial s}\left(\frac{1}{\rho_0}\right)\right\}\right]$$

$$I_0(s) = I_0(s_0)\left(\frac{\rho_0(s)c(s)}{\Delta A(s)}\frac{\Delta A(s_0)}{\rho_0(s_0)c(s_0)}\right)^{1/2}$$

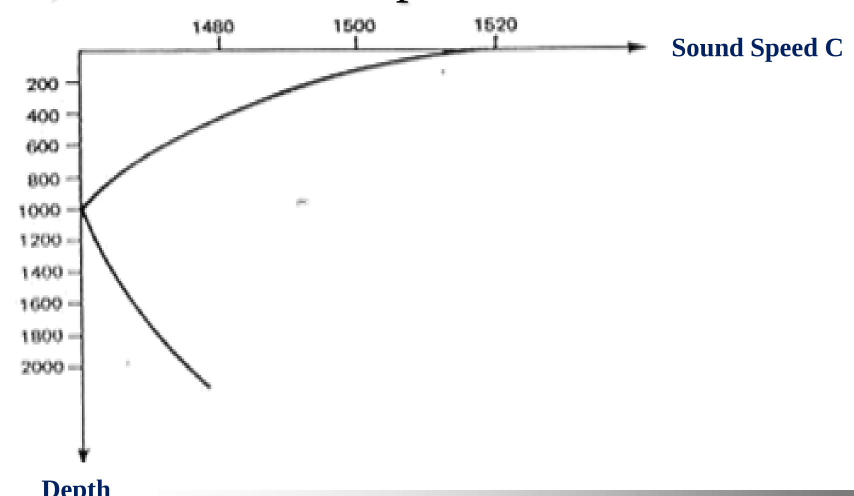
$$\left(\frac{I_0^2(s)A(s)}{\rho_0(s)c(s)}\right) = \text{constant along the ray tube}$$

Energy Flux Conserved

Underwater sound propagation

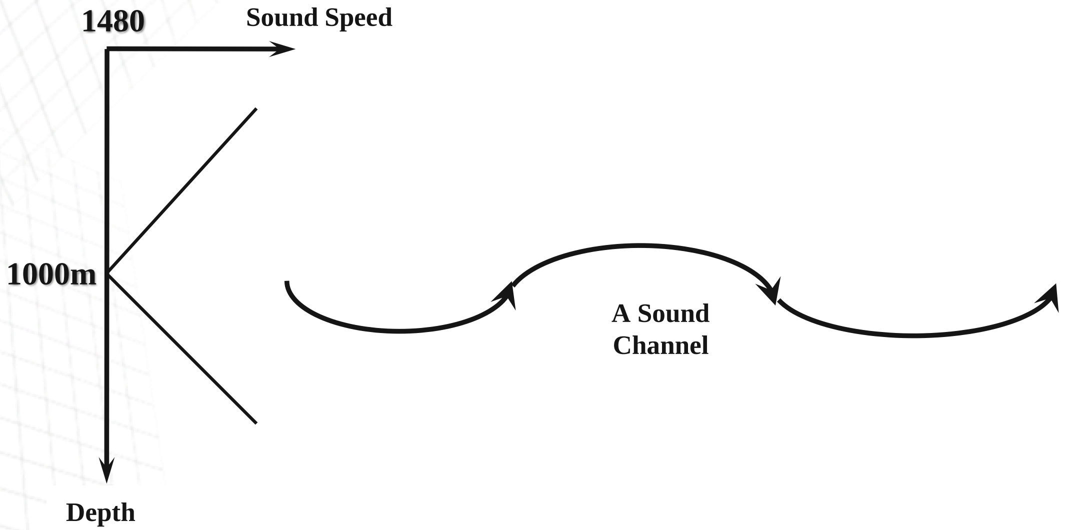
Sound waves of underwaters

- Water transmits sound waves far better than it does optical, radio or magnetic waves, and so sound is used extensively for underwater communication
- The temperature near the surface is warm, and temperature is decreased also to 1000m. It means that speed of sound is also decreases
- Under the 1000m, however, the increase of pressure leads to an increase of sound speed



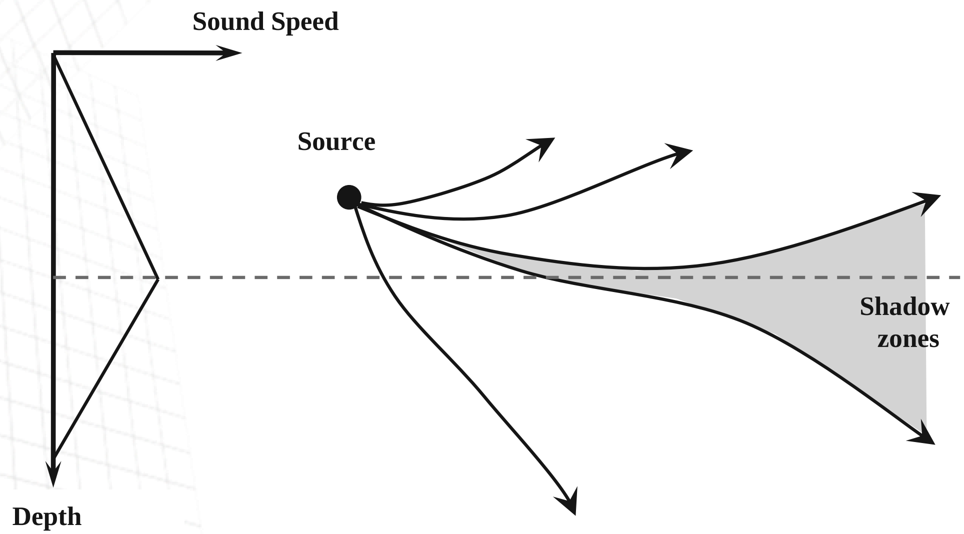
❖ Underwater sound propagation

- The rays propagating downwards will eventually reaches the regions which speed of sound is changed inversely, and the rays are propagating upwards. Finally the rays are trapped within the region. This regions are called '*sound channel*'



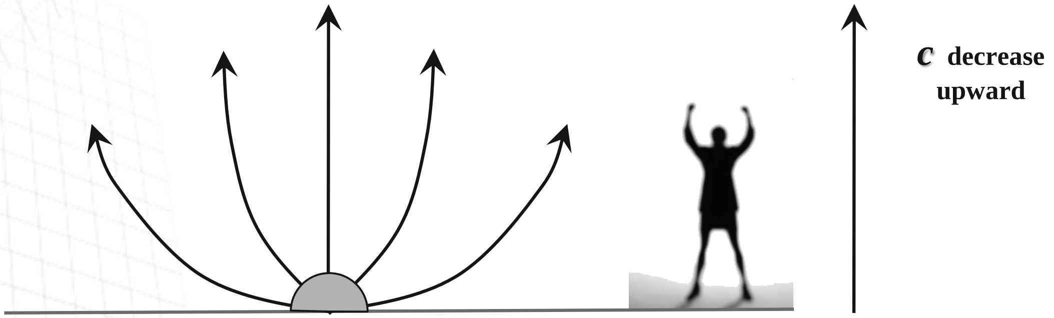
❖ Underwater sound propagation

- The refraction of sound rays by anomalies of the sound speed profile in the ocean can lead to the formation of *'shadow zones'*
- In the case of a sound source in the ocean near a position of maximum in the sound velocity, rays propagating upwards move into regions where the sound speed decrease and are refracted upwards, and vice versa.



❖ Sound propagation in the atmosphere

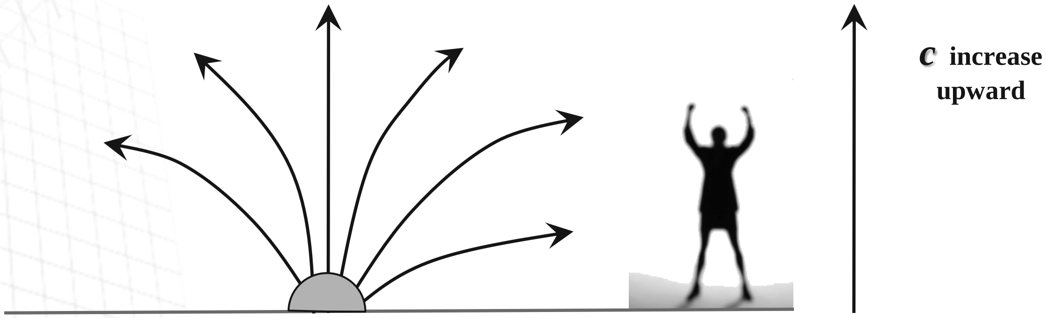
- Sound refraction on a day time
  - During the daytime the air temperature tends to decrease with height above earth, and hence the sound speed decrease upwards. Rays are then bent up, and most of the sound will pass over the head of a distant observer



Sound propagation on a typical day

❖ Sound propagation in the atmosphere

- Sound refraction on a night time
  - Sometimes on a clear night, the ground cools more quickly than the air, then rays are bent back to the ground are heard more distinctly by an observer



Sound propagation on a clear night following a warm day